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On the equilibrium of piezoelectric bodies of revolution

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Abstract

By means of the three-dimensional general solution in displacement functions (weighted harmonic functions) for piezoelectric materials, the general solution of stress components and electric displacements expressed by the displacement functions is derived by use of the constitutive relation and the equilibrium equations. Based on this general solution, a series of problems is solved by the trial-and-error method, including circular plate (or cylinder), annular plate (or hollow cylinder), cone and hollow cone. These problems are circular plates and cylinders under uniform radial or axial tension and electric displacements as well as pure bending, simply-supported circular plates subjected to uniformly distributed loads, rotating disks and circular shafts, cones or hollow cones subjected to concentrated forces plus charge and concentrated force couple at their apex, etc. Analytical solutions to various problems are obtained. When the cone apex angle 2α equals π, the solutions for the cases of concentrated forces plus point charges and torsion reduce to the simple and practical solutions of the half-space problem.

1. Introduction

Both class 6 mm crystals and piezoelectric ceramics of similar crystal symmetry belong to transversely isotropic piezoelectric material. Due to its excellent piezoelectric properties, it has found widespread applications. Therefore, it is necessary to make theoretical analysis and accurate quantitative descriptions of electric and stress fields inside piezoelectric ceramic components in the working condition caused by the joint action of mechanical loads and electric fields, from the point of view of electro-mechanical coupling. There is a series of classical problems concerning the body of revolution in the theory of elasticity as shown in Timoshenko and Goodier (1970) and Love (1994), including circular plates or cylinders under uniform axial and radial tension (or compression) and pure bending, simply-supported circular plates subjected to uniformly-distributed loads and uniformly rotating circular shafts and disks, etc. Love (1994) and Luré (1964) reported the solutions of the problem of a cone subjected to concentrated forces at its apex, while there is little study on the annular plate, the hollow cylinder and

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the hollow cone. As for transversely isotropic materials, Lekhniskii (1981) and Hu (1953) studied the problem of plate bending and the bending and compression problems of a cone subjected to concentrated forces at its apex. Chen (1965) studied the bending problem of a hollow cone subjected to concentrated forces at its apex, in addition to a solid cone problem. Furthermore, Ding et al. (1995) investigated the compression, bending and torsion problems of a spherically isotropic cone subjected to concentrated forces and force couples.

In regard to piezoelectric materials, Kogan et al. (1996) gave an analytical solution of infinite body with spheroidal inclusion under the joint action of uniform loads, electric displacement, in-plane shear and off-plane shear. Lee and Jiang (1996) made an accurate three-dimensional analysis of a simply-supported rectangular piezoelectric plate by state space approach. Ding et al. (1996b) transformed the basic equations for the case of a distributed body force and a body electric charge into a series of volume potential problems. A closed-form fundamental solution for the case of characteristics roots $s_1 \neq s_2 \neq s_3 \neq s_1$ was obtained by means of integration, which is of simple form. Dunn and Wienecke (1996) also gave the closed-form fundamental solution for characteristics roots $s_1 \neq s_2 \neq s_3 \neq s_1$ using the general solution and trial-and-error method. Ding et al. (1997a), by use of a simpler general solution and the trial-and-error method, gave the fundamental solutions for all cases of characteristic roots ($s_1 \neq s_2 \neq s_3 \neq s_1$, $s_1 \neq s_2 = s_3$, $s_1 = s_2 = s_3$) and Green's function for semi-infinite body and two-phase material. Sosa and Castro (1994) presented the solutions for the cases of concentrated loads and point charge applied at the line boundary of a piezoelectric half-plane. Ding et al. (1997b) obtained the solutions for a piezoelectric wedge subjected to concentrated forces and point charge. Ding et al. (1996a) also gave the solution of concentrated forces applied at the boundary of a piezoelectric half-plane, which was derived by the Fourier transform.

In this paper, the equilibrium of two important classes of piezoelectric body of revolution—circular plate (or cylinder) and cone, is systematically studied and a series of analytical solutions is acquired, which includes circular plates or cylinders under uniform axial and radial tension plus uniform electric displacements and pure bending, simply-supported circular plates subjected to uniformly distributed loads, rotating disks and circular shafts, as well as cones or hollow cones subjected to concentrated forces plus point charge and concentrated force couple at their apex, etc. When the apex angle of the cone 2α is π, the solutions for concentrated forces plus point charge and torsion are able to reduce to the solutions of the half-space problem, which are simple in form and easy to verify and utilize. In the following process of solution, as for the circular plate and cylinder problems, the solutions for the annular plate and the hollow cylinder are first deduced, then these solutions are reduced to the solutions for a solid circular plate and a cylinder. With respect to the cone and hollow cone problems, the cone problem is studied first, then the method will be extended to the hollow cone problem.

2. General solution to the problem of the piezoelectric body of revolution

As suggested by Sosa and Castro (1993), the governing equations for the theory of piezoelectricity are:

$$\sigma_{ij,j} = -F_i + \rho \frac{\partial^2 u_i}{\partial t^2}$$

$$D_{j,j} = \rho_f$$

$$\sigma_{ij} = C_{ijkl}\bar{\varepsilon}_{kl} - e_{kij}E_k$$

$$D_i = e_{ikl}\bar{\varepsilon}_{kl} + \varepsilon_{ik}E_k$$

$$\bar{\varepsilon}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$E_i = -\Phi_{,i} \tag{1}$$

where σ_{ij}, $\bar{\varepsilon}_{ij}$, u_i, E_i and D_i are the components of stress, strain, displacement, electric field and electric displacement, respectively; Φ is the electrical potential; ρ, F_i, ρ_f are material density, body force and density of free charges, respectively; and c_{ijkl}, e_{kij} and ε_{ij} are the elastic stiffness, piezoelectric and dielectric constants, respectively. In the most general case of anisotropy, there are altogether 45 independent constants. The present study is concerned, in particular, with the transversely isotropic piezoelectric materials as they represent what is possibly the most technologically important piezoelectric material. Thus, only 10 independent material constants are present. Removing the inertia terms in the dynamic equations in eqn (1), we get the corresponding equilibrium equations.

Ding et al. (1996a) gave the general solution in terms of four displacement functions for the dynamic equations for transversely isotropic piezoelectric media. For the equilibrium equations and in the case of characteristic roots $s_1 \neq s_2 \neq s_3 \neq s_1$, the solution takes the following form in cylindrical coordinates

$$u_r = \sum_{i=1}^{3} \frac{\partial \psi_i}{\partial r} - \frac{1}{r}\frac{\partial \psi_0}{\partial \theta}, \quad w = \sum_{i=1}^{3} s_i k_{1i} \frac{\partial \psi_i}{\partial z_i}$$

$$u_\theta = \sum_{i=1}^{3} \frac{1}{r}\frac{\partial \psi_i}{\partial \theta} + \frac{\partial \psi_0}{\partial r}, \quad \Phi = \sum_{i=1}^{3} s_i k_{2i} \frac{\partial \psi_i}{\partial z_i} \tag{2}$$

where $z_i = s_i z$ $(i = 0, 1, 2, 3)$ and $s_0 = \sqrt{c_{66}/c_{44}}$, s_i $(i = 1, 2, 3)$ are the three characteristic roots of a sixth-degree equation defined in Ding et al. (1996) and satisfy Re $(s_i) > 0$, k_{1i} and k_{2i} are constants dependent on materials constants and characteristic roots and the displacement functions ψ_i satisfy the following equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial \theta^2} + \frac{\partial^2}{\partial z_i^2}\right)\psi_i = 0, \quad (i = 0, 1, 2, 3) \tag{3}$$

Using the constitutive relation and eqn (2), the general solution of stresses and electric displacements expressed by four displacement functions are obtained. At this point, the coefficients in front of the derivatives of the displacement functions with respect to coordinates are all products or linear combinations of material constants and characteristic roots. If expressions of the stresses and electric displacements are substituted into the equilibrium and Gauss equations in the absence of F_i and ρ_f, some relations among these coefficients will be determined with consideration of eqn (3). With these relations being taken into account, the general solutions of stress components and electric displacements can be written as follows:

$$\sigma_r = 2c_{66} \sum_{i=1}^{3} \frac{\partial^2 \psi_i}{\partial r^2} + \sum_{i=1}^{3} m_i \frac{\partial^2 \psi_i}{\partial z_i^2} - 2c_{66}\frac{\partial}{\partial r}\left(\frac{\partial \psi_0}{r\partial \theta}\right) \tag{4}$$

or

$$\sigma_r = -2c_{66}\sum_{i=1}^{3}\left(\frac{\partial}{r\partial r}+\frac{\partial^2}{r^2\partial\theta^2}\right)\psi_i + \sum_{i=1}^{3} n_i\frac{\partial^2\psi_i}{\partial z_i^2} - 2c_{66}\frac{\partial}{\partial r}\left(\frac{\partial\psi_0}{r\partial\theta}\right)$$

$$\sigma_\theta = 2c_{66}\sum_{i=1}^{3}\left(\frac{\partial}{r\partial r}+\frac{\partial^2}{r^2\partial\theta^2}\right)\psi_i + \sum_{i=1}^{3} m_i\frac{\partial^2\psi_i}{\partial z_i^2} + 2c_{66}\frac{\partial}{\partial r}\left(\frac{\partial\psi_0}{r\partial\theta}\right)$$

$$\sigma_z = \sum_{i=1}^{3} a_i\frac{\partial^2\psi_i}{\partial z_i^2},\ D_z = \sum_{i=1}^{3} c_i\frac{\partial^2\psi_i}{\partial z_i^2}$$

$$\tau_{\theta z} = \sum_{i=1}^{3} s_i a_i\frac{\partial^2\psi_i}{r\partial\theta\partial z_i} + s_0 c_{44}\frac{\partial^2\psi_0}{\partial r\partial z_0}$$

$$D_\theta = \sum_{i=1}^{3} s_i c_i\frac{\partial^2\psi_i}{r\partial\theta\partial z_i} + s_0 e_{15}\frac{\partial^2\psi_0}{\partial r\partial z_0}$$

$$\tau_{zr} = \sum_{i=1}^{3} s_i a_i\frac{\partial^2\psi_i}{\partial r\partial z_i} - s_0 c_{44}\frac{\partial^2\psi_0}{r\partial\theta\partial z_0}$$

$$D_r = \sum_{i=1}^{3} s_i c_i\frac{\partial^2\psi_i}{\partial r\partial z_i} - s_0 e_{15}\frac{\partial^2\psi_0}{r\partial\theta\partial z_0}$$

$$\tau_{r\theta} = 2c_{66}\sum_{i=1}^{3}\frac{\partial}{\partial r}\left(\frac{\partial\psi_i}{r\partial\theta}\right) + c_{66}\left(2\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z_0^2}\right)\psi_0$$

where

$$a_i = s_i^2(c_{33}k_{1i}+e_{33}k_{2i}) - c_{33}, \quad n_i = -a_i s_i^2$$

$$c_i = e_{15}(1+k_{1i}) - \varepsilon_{11}k_{2i}, \quad m_i = n_i + 2c_{66} \quad (i = 1, 2, 3) \tag{5}$$

In regard to axisymmetric problems, let ψ_0 and ψ_i ($i = 1, 2, 3$) in eqns (3)–(5) be independent of θ.

In the later sections, various equilibrium problems are studied by use of harmonic polynomials and harmonic functions presented in Appendix A. In Sections 3–7, axisymmetric deformation problems are studied, hence, ψ_0 is taken to be zero. Let the z-axis be the symmetric axis and origin O be in the middle plane of the plate and cylinder.

Because displacement functions ψ_j satisfy weighted harmonic eqn (3), all harmonic functions in Appendix A can be chosen as displacement functions simply by replacing z with z_i just as illustrated in the following sections. There are so many symbols in this paper that a Nomenclature is introduced in Appendix D for convenient reference.

3. Rigid body displacements and identical electric potential

Using $\varphi_1\ (r, z)$ in eqn (A2) in Appendix A, we constitute the displacement function

$$\psi_i = A_{1i}\varphi_1(r, z_i) = A_{1i}z_i \tag{6}$$

where $A_{1i}\ (i = 1, 2, 3)$ are unknown constants to be determined.

Substituting eqn (6) into eqns (2) and (4) gives

$$u_r = 0, \quad w = \sum_{i=1}^{3} s_i k_{1i} A_{1i}, \quad \Phi = \sum_{i=1}^{3} s_i k_{2i} A_{1i} \tag{7}$$

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0, \quad D_r = D_z = 0 \tag{8}$$

The above equations contain two physically sensible solutions. One is that a piezoelectric body of revolution may have rigid body displacement w_0, that is,

$$u_r = 0, \quad w = w_0, \quad \Phi = 0, \quad \sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0, \quad D_r = D_z = 0 \tag{9}$$

The other is that a piezoelectric body of revolution may have identical potential Φ_0, i.e.

$$u_r = w = 0, \quad \Phi = \Phi_0, \quad \sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0, \quad D_r = D_z = 0 \tag{10}$$

4. Uniform axial tension, radial compression and axial electrical displacement

Using $\varphi_2(r, z)$ in eqn (A2) and $\gamma_0(r, z)$ in eqn (A5) in Appendix A, we constitute the displacement function

$$\psi_i = A_{2i}\varphi_2(r, z_i) + B_0\gamma_0(r, z_i) = A_{2i}\left(z_i^2 - \frac{r^2}{2}\right) + B_0 \ln\frac{r}{r_1} \tag{11}$$

where $A_{2i}\,(i = 1, 2, 3)$ and B_0 are unknown constants to be determined.

Substituting eqn (11) into eqns (2) and (4) results in

$$u_r = -\sum_{i=1}^{3} A_{2i}r + B_0\frac{1}{r}, \quad w = 2\sum_{i=1}^{3} s_i k_{1i} A_{2i} z_i, \quad \Phi = 2\sum_{i=1}^{3} s_i k_{2i} A_{2i} z_i \tag{12}$$

$$\sigma_r = 2\sum_{i=1}^{3} e_i A_{2i} - 2c_{66}B_0\frac{1}{r^2}, \quad \sigma_\theta = 2\sum_{i=1}^{3} e_i A_{2i} + 2c_{66}B_0\frac{1}{r^2}$$

$$\sigma_z = 2\sum_{i=1}^{3} a_i A_{2i}, \quad \tau_{rz} = 0, \quad D_r = 0, \quad D_z = 2\sum_{i=1}^{3} c_i A_{2i} \tag{13}$$

where

$$e_i = m_i - c_{66} \tag{14}$$

Eqns (12) and (13) contain three physically sensible solutions, that is, solutions for an annular plate or a hollow cylinder subjected to uniform axial tension and radial compression and uniform axial electric displacement. The boundary conditions are

$$r = r_k:\quad \begin{cases} \sigma_r = q_k \\ \tau_{rz} = 0 \quad (k = 0,\ 1), \\ D_r = 0 \end{cases} \quad z = \pm\frac{h}{2}:\quad \begin{cases} \sigma_z = p \\ \tau_{rz} = 0 \\ D_z = d_2 \end{cases} \tag{15}$$

Substituting eqn (13) into eqn (15) leads to

$$2r_0^2\sum_{i=1}^{3} e_i A_{2i} - 2c_{66}B_0 + r_0^2 q_0,\quad 2r_1^2\sum_{i=1}^{3} e_i A_{2i} - 2c_{66}B_0 = r_1^2 q_1,\quad 2\sum_{i=1}^{3} a_i A_{2i} = p,\quad 2\sum_{i=1}^{3} c_i A_{2i} = d_2 \tag{16}$$

4.1. Uniform radial compression

When $p = 0$ and $d_2 = 0$, eqn (16) gives the following solution

$$A_{21} = \frac{D_{11}}{\delta_1},\quad A_{22} = \frac{D_{12}}{\delta_1},\quad A_{23} = \frac{D_{13}}{\delta_1},\quad B_0 = l_1 \tag{17}$$

where

$$\delta_1 = e_1(a_2c_3 - a_3c_2) + e_2(a_3c_1 - a_1c_3) + e_3(a_1c_2 - a_2c_1)$$

$$D_{11} = l_2(a_2c_3 - a_3c_2),\quad D_{12} = l_2(a_3c_1 - a_1c_3),\quad D_{13} = l_2(a_1c_2 - a_2c_1)$$

$$l_1 = \frac{r_0^2 r_1^2(q_1 - q_0)}{2c_{66}\left(r_1^2 - r_0^2\right)},\quad l_2 = \frac{r_1^2 q_1 - r_0^2 q_0}{2\left(r_1^2 - r_0^2\right)} \tag{18}$$

The solutions for an annular plate or a hollow cylinder under uniform radial compression can be obtained by substituting eqn (17) into eqns (12) and (13), in which stress components and electric displacements are

$$\sigma_r = \frac{r_1^2 q_1 - r_0^2 q_0}{r_1^2 - r_0^2} + \frac{(q_0 - q_1)r_0^2 r_1^2}{r_1^2 - r_0^2}\frac{1}{r^2}$$

$$\sigma_\theta = \frac{r_1^2 q_1 - r_0^2 q_0}{r_1^2 - r_0^2} - \frac{(q_0 - q_1)r_0^2 r_1^2}{r_1^2 - r_0^2}\frac{1}{r^2}$$

$$\sigma_z = \tau_{rz} = 0,\quad D_r = D_z = 0 \tag{19}$$

Putting $r_0 = 0$ in eqns (18) and (19) gives the solutions for circular plate and cylinder.

4.2. Uniform axial tension

When $q_1 = q_1 = 0$ and $d_2 = 0$, from eqn (16) we have

$$A_{21} = \frac{D_{21}}{\delta_2}, \quad A_{22} = \frac{D_{22}}{\delta_2}, \quad A_{23} = \frac{D_{23}}{\delta_2}, \quad B_0 = 0 \tag{20}$$

where

$$\delta_2 = a_1(e_2c_3 - e_3c_2) + a_2(e_3c_1 - e_1c_3) + a_3(e_1c_2 - e_2c_1)$$

$$D_{21} = \frac{p(e_2c_3 - e_3c_2)}{2}, \quad D_{22} = \frac{p(e_3c_1 - e_1c_3)}{2}, \quad D_{23} = \frac{p(e_1c_2 - e_2c_1)}{2} \tag{21}$$

Substituting eqn (20) into eqns (12) and (13) gives the solution for a circular plate or a cylinder under uniform axial tension and compression, in which stress components and electric displacements are

$$\sigma_z = p, \quad \sigma_r = \sigma_\theta = \tau_{rz} = 0$$

$$D_r = D_z = 0 \tag{22}$$

4.3. Uniform axial electric displacement

When $q_0 = q_1 = 0$ and $p = 0$, solving eqn (16) leads to

$$A_{21} = \frac{D_{31}}{\delta_3}, \quad A_{22} = \frac{D_{32}}{\delta_3}, \quad A_{23} = \frac{D_{33}}{\delta_3}, \quad B_0 = 0 \tag{23}$$

where

$$\delta_1 = c_1(a_2e_3 - a_3e_2) + c_2(a_3e_1 - a_1e_3) + c_3(a_1e_2 - a_2e_1)$$

$$D_{11} = \frac{d_2(a_2e_3 - a_3e_2)}{2}, \quad D_{12} = \frac{d_2(a_3e_1 - a_1e_3)}{2}, \quad D_{13} = \frac{d_2(a_1e_2 - a_2e_1)}{2} \tag{24}$$

The solutions for a circular plate or a cylinder under uniform axial electric displacement can be obtained by substituting eqn (23) into eqns (12) and (13), in which stresses and electric displacements are

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0$$

$$D_r = 0, \quad D_z = d_2 \tag{25}$$

Eqns (19), (22) and (25) show solutions of stresses and electric displacement are independent of material constants (elastic constants, piezoelectric constants and dielectric constants). Furthermore, eqns (22) and (25) even show that the solutions are independent of geometric dimensions and shape.

5. Pure bending and uniform radial electric displacement

Using $\varphi_3(r, z)$ in eqn (A2) and $\gamma_1(r, z)$ in eqn (A5) in Appendix A, we constitute the displacement function

$$\psi_i = A_{3i}\varphi_3(r, z_i) + B_{1i}\gamma_1(r, z_i) = A_{3i}\left(z_i^3 - \frac{3r^2 z_i}{2}\right) + B_{1i} z_i \ln\frac{r}{r_1} \tag{26}$$

where A_{3i} and B_{1i} ($i = 1, 2, 3$) are unknown constants to be determined.

Substituting eqn (26) into eqns (2) and (4) gives

$$u_r = -3\sum_{i=1}^{3} A_{3i} r z_i + \sum_{i=1}^{3} B_{1i}\frac{z_i}{r}$$

$$w = 3\sum_{i=1}^{3} s_i k_{1i} A_{3i}\left(z_i^2 - \frac{1}{2}r^2\right) + \sum_{i=1}^{3} s_i k_{1i} B_{1i} \ln\frac{r}{r_1}$$

$$\Phi = 3\sum_{i=1}^{3} s_i k_{2i} A_{3i}\left(z_i^2 - \frac{1}{2}r^2\right) + \sum_{i=1}^{3} s_i k_{2i} B_{1i} \ln\frac{r}{r_1} \tag{27}$$

$$\sigma_r = 6\sum_{i=1}^{3} e_i A_{3i} z_i - 2c_{66}\sum_{i=1}^{3} B_{1i}\frac{z_i}{r^2}$$

$$\sigma_\theta = 6\sum_{i=1}^{3} e_i A_{3i} z_i + 2c_{66}\sum_{i=1}^{3} B_{1i}\frac{z_i}{r^2}$$

$$\sigma_z = 6\sum_{i=1}^{3} a_i A_{3i} z_i, \quad \tau_{rz} = -3\sum_{i=1}^{3} s_i a_i A_{3i} r + \sum_{i=1}^{3} s_i a_i B_{1i}\frac{1}{r}$$

$$D_r = -3\sum_{i=1}^{3} s_i c_i A_{3i} r + \sum_{i=1}^{3} s_i c_i B_{1i}\frac{1}{r}, \quad D_z = 6\sum_{i=1}^{3} c_i A_{3i} z_i \tag{28}$$

The above equations contain two physically sensible solutions, that is, the solution for an annular plate or a hollow cylinder under pure bending or uniform radial electric displacement, as described below.

5.1. Pure bending

Boundary conditions are

$$r = r_k : \begin{cases} \tau_{rz} = 0 \\ \int_{-h/2}^{h/2} z\sigma_r \, \mathrm{d}z = M_k \quad (k = 0,1), \\ D_r = 0 \end{cases} \quad z = \pm\frac{h}{2} : \begin{cases} \tau_{rz} = 0 \\ \sigma_z = 0 \\ D_z = 0 \end{cases} \tag{29}$$

Substituting eqn (28) into eqn (29) gives

$$\sum_{i=1}^{3} s_i a_i A_{3i} = 0 \tag{30}$$

$$\sum_{i=1}^{3} s_i a_i B_{1i} = 0 \tag{31}$$

$$\sum_{i=1}^{3} s_i c_i A_{3i} = 0 \tag{32}$$

$$\sum_{i=1}^{3} s_i c_i B_{1i} = 0 \tag{33}$$

$$3r_0^2 \sum_{i=1}^{3} s_i e_i A_{3i} - c_{66} \sum_{i=1}^{3} s_i B_{1i} = \frac{6r_0^2 M_0}{h^3} \tag{34}$$

$$3r_1^2 \sum_{i=1}^{3} s_i e_i A_{3i} - c_{66} \sum_{i=1}^{3} s_i B_{1i} = \frac{6r_1^2 M_0}{h^3} \tag{35}$$

From eqns (34) and (35), we have

$$\sum_{i=1}^{3} s_i e_i B_{3i} = k_1 \tag{36}$$

$$\sum_{i=1}^{3} s_i B_{1i} = k_2 \tag{37}$$

where

$$k_1 = \frac{2\left(r_0^2 M_0 - r_1^2 M_1\right)}{h^3\left(r_0^2 - r_1^2\right)}, \quad k_2 = \frac{6r_0^2 r_1^2 (M_1 - M_0)}{c_{66} h^3 \left(r_1^2 - r_0^2\right)} \tag{38}$$

The unknown constants A_{3i} can be calculated from eqns (30), (32) and (36) and B_{1i} from eqns (31), (33) and (37).

$$A_{31}=\frac{s_2s_3k_1(a_2c_3-a_3c_2)}{\Delta_1},\quad A_{32}=\frac{s_3s_1k_1(a_3c_1-a_1c_3)}{\Delta_1},\quad A_{33}=\frac{s_1s_2k_1(a_1c_2-a_2c_1)}{\Delta_1}$$

$$\Delta_1=s_1s_2s_3\left[e_1(a_2c_3-a_3c_2)+e_2(a_3c_1-a_1c_3)+e_3(a_1c_2-a_2c_1)\right]$$

$$B_{11}=\frac{s_2s_3k_2(a_2c_3-a_3c_2)}{\Delta_2},\quad B_{12}=\frac{s_3s_1k_2(a_3c_1-a_1c_3)}{\Delta_2},\quad B_{13}=\frac{s_1s_2k_2(a_1c_2-a_2c_1)}{\Delta_2}$$

$$\Delta_2=s_1s_2s_3(a_2c_3-a_3c_2+a_3c_1-a_1c_3+a_1c_2-a_2c_1) \tag{39}$$

Substituting eqn (39) into eqns (27) and (28) leads to the solution for an annular plate under pure bending, in which stresses and electric displacements are

$$\sigma_r=\frac{12z}{h^3\left(r_1^2-r_0^2\right)}\left[r_1^2M_1-r_0^2M_0-\frac{r_0^2r_1^2(M_1-M_0)}{r^2}\right]$$

$$\sigma_\theta=\frac{12z}{h^3\left(r_1^2-r_0^2\right)}\left[r_1^2M_1-r_0^2M_0+\frac{r_0^2r_1^2(M_1-M_0)}{r^2}\right]$$

$$\sigma_z=\tau_{rz}=0,\quad D_r=D_z=0 \tag{40}$$

5.2. Uniform radial electric displacement

The boundary conditions are

$$r=r_k:\begin{cases}\sigma_r=0\\ \tau_{rz}=0\quad(k=0,1),\\ D_r=d_k\end{cases}\quad z=\pm\frac{h}{2}:\begin{cases}\sigma_z=0\\ \tau_{rz}=0\\ D_z=\pm\dfrac{(r_0d_0-r_1d_1)h}{r_1^2-r_0^2}\end{cases} \tag{41}$$

Substituting eqn (28) into eqn (41) leads to

$$\sum_{i=1}^{3}s_ia_iA_{3i}=0 \tag{42}$$

$$\sum_{i=1}^{3}s_ia_iB_{1i}=0 \tag{43}$$

$$\sum_{i=1}^{3}s_ie_iA_{3i}=0 \tag{44}$$

$$\sum_{i=1}^{3} s_i B_{1i} = 0 \tag{45}$$

$$-3r_0^2 \sum_{i=1}^{3} s_i c_i A_{3i} + \sum_{i=1}^{3} s_i c_i B_{1i} = r_0 d_0 \tag{46}$$

$$-3r_1^2 \sum_{i=1}^{3} s_i c_i A_{3i} + \sum_{i=1}^{3} s_i c_i B_{1i} = r_1 d_1 \tag{47}$$

From eqns (46) and (47), we have

$$\sum_{i=1}^{3} s_i c_i A_{3i} = k_3 \tag{48}$$

$$\sum_{i=1}^{3} s_i c_i B_{1i} = k_4 \tag{49}$$

where

$$k_3 = \frac{r_0 d_0 - r_1 d_1}{3\left(r_1^2 - r_0^2\right)}, \quad k_4 = \frac{r_0 r_1 (r_1 d_0 - r_0 d_1)}{\left(r_1^2 - r_0^2\right)} \tag{50}$$

The unknown constants A_{3i} can be obtained from eqns (42), (44) and (48) and B_{1i} from eqns (43), (45) and (49).

$$A_{31} = \frac{s_2 s_3 k_3 (a_2 e_3 - a_3 e_2)}{\Delta_1}, \quad A_{32} = \frac{s_3 s_1 k_3 (a_2 e_1 - a_1 e_2)}{\Delta_1}, \quad A_{33} = \frac{s_1 s_2 k_3 (a_1 e_2 - a_2 e_1)}{\Delta_1}$$

$$B_{11} = \frac{s_2 s_3 k_4 (a_2 - a_3)}{\Delta_2}, \quad B_{12} = \frac{s_3 s_1 k_4 (a_3 - a_1)}{\Delta_2}, \quad B_{13} = \frac{s_1 s_2 k_4 (a_1 - a_2)}{\Delta_2} \tag{51}$$

Substituting eqn (51) into eqns (27) and (28) leads to the solution for an annular plate or a hollow cylinder under uniform radial electric displacement, in which stresses and electric displacements are

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0$$

$$D_r = \frac{r_1 d_1 - r_0 d_0}{r_1^2 - r_0^2} r + \frac{r_0 r_1 (r_1 d_0 - r_0 d_1)}{r_1^2 - r_0^2} \frac{1}{r}$$

$$D_z = \frac{2(r_0 d_0 - r_1 d_1)}{r_1^2 - {}_0^2} z \tag{52}$$

The solution for a circular plate or a cylinder can be obtained just by letting $r_0 = 0$ in the above solution. Similarly, from eqns (40) and (52) it can be seen that stresses and electric displacements are independent of material constants.

6. Annular plate simply-supported on outer and inner surfaces under uniform axial loads

Using $\varphi_3(r, z)$ and $\varphi_5(r, z)$ in eqn (A2) and $\gamma_1(r, z)$ and $\gamma_3(r, z)$ in eqn (A5) in Appendix A, we constitute the displacement function

$$\psi_i = F_{3i}\varphi_3(r, z_i) + F_{5i}\varphi_5(r, z_i) + G_{1i}\gamma_1(r, z_i) + G_{3i}\gamma_3(r, z_i)$$

$$= F_{3i}\left(z_i^3 - \frac{3r^2 z_i}{2}\right) + F_{5i}\left(z_i^5 - 5r^2 z_i^3 + \frac{15}{8}r^4 z_i\right)$$

$$+G_{1i}z_i \ln\frac{r}{r_1} + G_{3i}\left[\left(z_i^3 - \frac{3}{2}r^2 z_i\right)\ln\frac{r}{r_1} + \frac{3r^2 z_i}{2}\right] \tag{53}$$

where F_{3i}, F_{5i}, G_{1i} and G_{3i} ($i = 1, 2, 3$) are unknown constants to be determined.

Substituting eqn (53) into eqns (2) and (4) leads to

$$u_r = -3\sum_{i=1}^{3} F_{3i}z_i r - 5\sum_{i=1}^{3} F_{5i}\left(2rz_i^3 + \frac{3}{2}r^3 z_i\right) + \sum_{i=1}^{3} G_{1i}\frac{z_i}{r}$$

$$+\sum_{i=1}^{3} G_{3i}\left(-3z_i r \ln\frac{r}{r_1} + \frac{z_i^3}{r} + \frac{3}{2}rz_i\right)$$

$$w = 3\sum_{i=1}^{3} s_i k_{1i} F_{3i}\left(z_i^2 - \frac{1}{2}r^2\right) + 5\sum_{i=1}^{3} s_i k_{1i} F_{5i}\left(z_i^4 - 3r^2 z_i^2 + \frac{3}{8}r^4\right)$$

$$+3\sum_{i=1}^{3} s_i k_{1i} G_{1i} \ln\frac{r}{r_1} + \sum_{i=1}^{3} s_i k_{1i} G_{3i}\left[\left(z_i^2 - \frac{1}{2}r^2\right)\ln\frac{r}{r_1} + \frac{1}{2}r^2\right]$$

$$\Phi = 3\sum_{i=1}^{3} s_i k_{2i} F_{3i}\left(z_i^2 - \frac{1}{2}r^2\right) + 5\sum_{i=1}^{3} s_i k_{2i} F_{5i}\left(z_i^4 - 3r^2 z_i^2 + \frac{3}{8}r^4\right)$$

$$+\sum_{i=1}^{3} s_i k_{2i} G_{1i} \ln\frac{r}{r_1} + 3\sum_{i=1}^{3} s_i k_{2i} G_{3i}\left[\left(z_i^2 - \frac{1}{2}r^2\right)\ln\frac{r}{r_1} + \frac{1}{2}r^2\right] \tag{54}$$

$$\sigma_r = 6\sum_{i=1}^{3} e_i F_{3i} z_i + 5\sum_{i=1}^{3} F_{5i}\left(4e_i z_i^3 - 3f_i r^2 z_i\right)$$

$$-2c_{66}\sum_{i=1}^{3} G_{1i}\frac{z_i}{r^2} + \sum_{i=1}^{3} G_{3i}\left[6e_i z_i \ln\frac{r}{r_1} - c_{66}\left(\frac{2z_i^3}{r^2} + 3z_i\right)\right]$$

$$\sigma_\theta = 6\sum_{i=1}^{3} e_i F_{3i} z_i + 5\sum_{i=1}^{3} F_{5i}\left(4e_i z_i^3 - 3g_i r^2 z_i\right)$$

$$+2c_{66}\sum_{i=1}^{3} G_{1i}\frac{z_i}{r^2} + \sum_{i=1}^{3} G_{3i}\left[6e_i z_i \ln\frac{r}{r_1} + c_{66}\left(\frac{2z_i^3}{r^2} + 3z_i\right)\right]$$

$$\sigma_z = 6\sum_{i=1}^{3} a_i F_{3i} z_i + 10\sum_{i=1}^{3} a_i F_{5i}\left(2z_i^3 - 3r^2 z_i\right) + 6\sum_{i=1}^{3} a_i G_{3i} z_i \ln\frac{r}{r_1}$$

$$\tau_{rz} = -3\sum_{i=1}^{3} s_i a_i F_{3i} r - 15\sum_{i=1}^{3} s_i a_i F_{5i}\left(2rz_i^2 - \frac{1}{2}r^3\right)$$

$$+\sum_{i=1}^{3} s_i a_i G_{1i}\frac{1}{r} - 3\sum_{i=1}^{3} s_i a_i G_{3i}\left(r\ln\frac{r}{r_1} - \frac{z_i^2}{r} - \frac{r}{2}\right)$$

$$D_r = -3\sum_{i=1}^{3} s_i c_i F_{3i} r - 15\sum_{i=1}^{3} s_i c_i F_{5i}\left(2rz_i^2 - \frac{1}{2}r^3\right)$$

$$+\sum_{i=1}^{3} c_i s_i G_{1i}\frac{1}{r} - 3\sum_{i=1}^{3} c_i s_i G_{3i}\left(r\ln\frac{r}{r_1} - \frac{z_i^2}{r} - \frac{r}{2}\right)$$

$$D_z = 6\sum_{i=1}^{3} c_i F_{3i} z_i + 10\sum_{i=1}^{3} c_i G_{5i}\left(2z_i^3 - 3r^2 z_i\right) + 6\sum_{i=1}^{3} c_i G_{3i} z_i \ln\frac{r}{r_1} \tag{55}$$

where

$$f_i = 2e_i - c_{66},\ g_i = 2e_i + c_{66} \tag{56}$$

The boundary conditions of an annular plate, which is simply-supported on the inner and outer surfaces and is under uniform axial loads, are

$$r = r_k\colon \begin{cases} \int_{-h/2}^{h/2} z\sigma_r\,\mathrm{d}z = 0 \\ w|_{z=0} = 0 \\ \int_{-h/2}^{h/2} D_r\,\mathrm{d}z = 0 \end{cases} \quad (k = 0, 1), \quad z = \pm\frac{h}{2}\colon \begin{cases} \sigma_z = \pm\frac{p_1}{2} \\ \tau_{rz} = 0 \\ D_z = 0 \end{cases} \tag{57}$$

From $z = \pm(h/2)$: $\tau_{rz} = 0$, we arrive at

$$\sum_{i=1}^{3} s_i a_i G_{3i} = 0 \tag{58}$$

$$2\sum_{i=1}^{3} s_i a_i F_{3i} + 5h^2 \sum_{i=1}^{3} s_i^3 a_i F_{5i} = 0 \tag{59}$$

$$\sum_{i=1}^{3} s_i a_i F_{5i} = 0 \tag{60}$$

$$\sum_{i=1}^{3} s_i a_i G_{1i} + \frac{3h^2}{4} \sum_{i=1}^{3} s_i^3 a_i G_{3i} = 0 \tag{61}$$

Similarly, from $z = \pm(h/2)$: $D_z = 0$, then

$$\sum_{i=1}^{3} s_i c_i G_{3i} = 0 \tag{62}$$

$$3\sum_{i=1}^{3} s_i c_i F_{3i} + \frac{5h^2}{2} \sum_{i=1}^{3} s_i^3 c_i F_{5i} = 0 \tag{63}$$

$$\sum_{i=1}^{3} s_i c_i F_{5i} = 0 \tag{64}$$

From $z = \pm(h/2)$: $\sigma_2 = \pm(p/2)$, we have

$$3h\sum_{i=1}^{3} s_i a_i F_{3i} + \frac{5h^3}{2} \sum_{i=1}^{3} s_i^3 a_i F_{5i} = \frac{p_1}{2} \tag{65}$$

From eqns (59) and (65), the following equations can be obtained

$$\sum_{i=1}^{3} s_i a_i F_{3i} = \frac{p_1}{4h} \tag{66}$$

$$\sum_{i=1}^{3} s_i^3 a_i F_{5i} = -\frac{p_1}{10h^3} \tag{67}$$

F_{5i} can be worked out from eqns (60), (64) and (67).

From $r = r_k$: $\int_{-h/2}^{h/2} D_r \, \mathrm{d}z = 0$ ($k = 0$, 1), we have

$$
\begin{aligned}
&-3r_0\sum_{i=1}^{3}s_ic_iF_{3i}-\frac{5r_0h^2}{2}\sum_{i=1}^{3}s_i^3c_iF_{5i}-\frac{15r_0^3}{2}\sum_{i=1}^{3}s_ic_iF_{5i}+\frac{1}{r_0}\sum_{i=1}^{3}s_ic_iG_{1i}\\
&\quad+\frac{r_0}{3}\ln\frac{r_0}{r_1}\sum_{i=1}^{3}s_ic_iG_{3i}+\frac{h^2}{4r_0}\sum_{i=1}^{3}s_i^3c_iG_{3i}+\frac{3r_0}{2}\sum_{i=1}^{3}s_ic_iG_{3i}=0
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
&-3r_1\sum_{i=1}^{3}s_ic_iF_{3i}-\frac{5r_1h^2}{2}\sum_{i=1}^{3}s_i^3c_iF_{5i}-\frac{15r_1^3}{2}\sum_{i=1}^{3}s_ic_iF_{5i}+\frac{1}{r_1}\sum_{i=1}^{3}s_ic_iG_{1i}\\
&\quad+\frac{h_2}{4r_1}\sum_{i=1}^{3}s_i^3c_iG_{3i}+\frac{3r_1}{2}\sum_{i=1}^{3}s_ic_iG_{3i}=0
\end{aligned}
\tag{69}
$$

By use of eqns (62)–(64), eqns (68) and (69) can be simplified to

$$
\sum_{i=1}^{3}s_ic_iG_{1i}+\frac{h^2}{4}\sum_{i=1}^{3}s_i^3c_iG_{3i}=0
\tag{70}
$$

From $r=r_k$: $\int_{-h/2}^{h/2}z\sigma_r$, $\mathrm{d}z=0$ $(k=0,\ 1)$, we get

$$
\begin{aligned}
&r_0^2\left[6\sum_{i=1}^{3}s_ie_iF_{3i}+3\sum_{i=1}^{3}s_i\left(h^2s_i^2e_i-5r_0^2f_i\right)F_{5i}\right]\\
&\quad-2c_{66}\sum_{i=1}^{3}s_iG_{1i}+3\sum_{i=1}^{3}s_i\left[2e_ir_0^2\ln\frac{r_0}{r_1}-c_{66}\left(\frac{h^2}{10}s_i^2+r_0^2\right)\right]G_{3i}=0
\end{aligned}
\tag{71}
$$

$$
r_1^2\left[6\sum_{i=1}^{3}s_ie_iF_{3i}+3\sum_{i=1}^{3}s_i\left(h^2s_i^2e_i-5r_0^2f_i\right)F_{5i}\right]-2c_{66}\sum_{i=1}^{3}s_iG_{1i}-3c_{66}\sum_{i=1}^{3}s_i\left(\frac{h^2}{10}s_i^2+r_1^2\right)G_{3i}=0
\tag{72}
$$

Superposing the rigid body solution eqn (9) on the equations above and from $r=r_k$, $z=0$: $w=0$ $(k=0,\ 1)$, we have

$$
\begin{aligned}
&r_0\left[2w_0-3r_0^2\sum_{i=1}^{3}s_ik_{1i}F_{3i}+\frac{15}{4}r_0^4\sum_{i=1}^{3}s_ik_{1i}F_{5i}+2\ln\frac{r_0}{r_1}\sum_{i=1}^{3}s_ik_{1i}G_{1i}-3r_0^2\left(\ln\frac{r_0}{r_1}-1\right)\sum_{i=1}^{3}s_ik_{1i}G_{3i}\right]\\
&\quad=0
\end{aligned}
\tag{73}
$$

$$
2w_0-3r_1^2\sum_{i=1}^{3}s_ik_{1i}F_{3i}+\frac{15}{4}r_1^4\sum_{i=1}^{3}s_ik_{1i}F_{5i}+3r_1^2\sum_{i=1}^{3}s_ik_{1i}G_{3i}=0
\tag{74}
$$

where w_0 is a constant to be determined.

Finally, w_0, F_{3i}, G_{1i} and G_{3i} can be determined from eqns (58), (61)–(63), (66) and (70)–(74) (ten equations altogether). Substituting F_{3i}, G_{1i}, G_{3i} and previously obtained F_{5i} back into eqns (54) and (55) yields the solution for an annular plate that is simply-supported on the outer and inner surfaces and uniformly loaded on the upper and lower surfaces with $\pm(p_1/2)$, respectively.

Table 1

Elastic constants (10^{10} Nm^{-2})					Piezoelectric constants (Cm^{-2})			Dielectric constants (10^{-10} CV^{-1} m^{-1})	
c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	e_{31}	e_{33}	e_{15}	ε_{11}	ε_{33}
12.6	7.78	7.43	11.5	2.56	-5.2	15.1	12.7	64.9	56.2

Table 2

Type of material	Thickness of circular plate (m)	Deflection at center (m)	Moment at center (Nm)
PZT-4	0.01	$6.2469 \times 10^{-6} p_1$	$0.2220 p_1$
	0.1	$6.2966 \times 10^{-9} p_1$	$0.2220 p_1$
Transversely isotropic material	0.01	$9.3627 \times 10^{-6} p_1$	$0.2114 p_1$
	0.1	$9.4493 \times 10^{-9} p_1$	$0.2114 p_1$

Superposing this solution on the solution of an annular plate under axial uniform tension as discussed in Section 4 and letting $p = (p_1/2)$ result in the solution for an annular plate that is simply-supported on the inner and outer surfaces and is loaded with uniform load p_1 on the upper surface. Let $r_0 = 0$, $G_{1i} = G_{3i} = 0$, then w_0 and F_{3i} can be derived from eqns (63), (66), (72) and (74). Thus, with w_0, F_{3i} and F_{5i} being known, the solution for a circular plate is obtained.

Assume that the thickness of a PZT-4 solid piezoelectric ceramic circular plate $h = 0.01$ or 0.1 m, radius $r_1 = 1$ m. The material constants of PZT-4 are shown in Table 1. Based on the equations above, the deflection and moment at the center of the circular plate can be calculated. Assume that a transversely isotropic circular plate has the same elastic constants as those of PZT-4 and the same geometric dimensions and boundary conditions. The results of the calculations are listed in Table 2 for comparison. It is obvious that the deflections at the center caused by uniform load p_1 on the upper surface are different, whereas the moments exhibit no noticeable difference.

7. Piezoelectric rotating circular shaft and rotating disk

7.1. Particular solution for the case of potential body force

In cylindrical coordinates, the fundamental equations of axisymmetric deformation can be expressed as three second-order partial differential equations in three unknown variables, i.e., displacements u_r, w and electric potential Φ. When the body force is potential, namely,

$$F_r = -\frac{\partial V}{\partial r}, \quad F_z = -\frac{\partial V}{\partial z} \tag{75}$$

then it can be assumed that

$$u_r = \frac{\partial U}{\partial r}, \quad w = \frac{\partial W}{\partial z}, \quad \Phi = \frac{\partial \Omega}{\partial z} \tag{76}$$

Substituting eqns (75) and (76) into the three second-order partial differential equations, the following equations can readily be derived

$$D\begin{Bmatrix} U \\ W \\ \Omega \end{Bmatrix} = \begin{Bmatrix} V \\ V \\ 0 \end{Bmatrix} \tag{77}$$

where U, W and Ω are functions to be determined, V is the potential of body force and D is a differential operator as described below,

$$D = \begin{bmatrix} c_{11}\Lambda + c_{44}\dfrac{\partial^2}{\partial z^2} & (c_{13}+c_{44})\dfrac{\partial^2}{\partial z^2} & (e_{31}+e_{15})\dfrac{\partial^2}{\partial z^2} \\ (c_{13}+c_{44})\Lambda & c_{44}\Lambda + c_{33}\dfrac{\partial^2}{\partial z^2} & e_{15}\Lambda + e_{33}\dfrac{\partial^2}{\partial z^2} \\ (e_{31}+e_{15})\Lambda & e_{15}\Lambda + e_{33}\dfrac{\partial^2}{\partial z^2} & -\left(\varepsilon_{11}\Lambda + \varepsilon_{33}\dfrac{\partial^2}{\partial z^2}\right) \end{bmatrix} \tag{78}$$

and where $\Lambda=(\partial^2/\partial r^2)+(1/r)(\partial/\partial r)$. The centrifugal force for uniform rotation is

$$F_r = \rho r\omega^2, \quad F_z = 0 \text{ and } V = -\frac{1}{2}\rho r^2\omega^2 \tag{79}$$

At this point, V in the second row of eqn (77) equals zero. Introduce a function F such that $|D|F = V$, where $|D|$ is the determinant of D. After some manipulations, we arrive at

$$\left(a\frac{\partial^6}{\partial z^6} + b\Lambda\frac{\partial^4}{\partial z^4} + c\Lambda^2\frac{\partial^2}{\partial z^2} + d\Lambda^3\right)F = \frac{1}{2}\rho r^2\omega^2 \tag{80}$$

where a, b, c and d are constants dependent on material constants (Ding et al., 1996a). It is easy to find a particular solution of F.

$$F = \frac{\rho\omega^2}{8^2 6^2 4^2 2d} r^8 \tag{81}$$

Obviously the particular solution of eqn (77) is

$$\begin{Bmatrix} U \\ W \\ \Omega \end{Bmatrix} = \begin{Bmatrix} A_{11}F \\ A_{12}F \\ A_{13}F \end{Bmatrix} \tag{82}$$

where A_{11}, A_{12} and A_{13} are algebraic complements of the first row in D. From eqns (76), (81) and (82), the particular solution of displacements and electric potential can be found to be

$$u_r = -\frac{\rho\omega^2}{8c_{11}} r^3, \quad w = 0, \quad \Phi = 0 \tag{83}$$

Substituting eqn (83) into the constitutive relation yields the particular solution of stresses and electric displacements.

$$\sigma_r = h_1 r^2, \quad \sigma_\theta = h_2 r^2, \quad \sigma_z = h_3 r^2, \quad \tau_{rz} = 0, \quad D_r = 0, \quad D_z = h_4 r^2 \tag{84}$$

where

$$h_1 = \frac{-(3c_{11}+c_{12})\rho\omega^2}{8c_{11}}, \quad h_2 = \frac{-(3c_{12}+c_{11})\rho\omega^2}{8c_{11}}, \quad h_3 = \frac{-c_{13}\rho\omega^2}{2c_{11}}, \quad h_4 = \frac{-e_{31}\rho\omega^2}{2c_{11}} \tag{85}$$

7.2. Solutions for a piezoelectric rotating disk

The boundary conditions of an annular piezoelectric rotating disk are

$$r = r_k: \begin{cases} \int_{-h/2}^{h/2} \sigma_r \, dz = 0 \\ \tau_{rz} = 0 \\ \int_{-h/2}^{h/2} D_r \, dz = 0 \end{cases} \quad (k = 0,1), \quad z = \pm\frac{h}{2}: \begin{cases} \sigma_z = 0 \\ \tau_{rz} = 0 \\ D_z = d_3 \end{cases} \tag{86}$$

Using $\varphi_2(r, z)$ and $\varphi_4(r, z)$ in eqn (A2) and $\gamma_o(r, z)$ in eqn (A5) in Appendix A, we constitute the displacement function

$$\begin{aligned} \psi_i &= F_{2i}\varphi_2(r, z_i) + F_{4i}\varphi_4(r, z_i) + G_0\gamma_0(r, z_i) \\ &= F_{2i}\left(z_i^2 - \frac{r^2}{2}\right) + F_{4i}\left(z_i^4 - 3r^2z_i^2 + \frac{3r^4}{8}\right) + G_0 \ln\frac{r}{r_1}, \quad (i = 1, 2, 3) \end{aligned} \tag{87}$$

where F_{2i}, F_{4i} ($i = 1, 2, 3$) and G_0 are unknown constants to be determined.

Substituting eqn (87) into eqns (2) and (4) leads to a solution. Superposing that solution with the particular solution represented by eqns (83) and (84) results in

$$\begin{aligned} u_r &= -\sum_{i=1}^{3} F_{2i}r + 3\sum_{i=1}^{3} F_{4i}\left(-2rz_i^2 + \frac{1}{2}r^3\right) + G_0\frac{1}{r} - \frac{\rho\omega^2}{8c_{11}}r^3 \\ w &= 2\sum_{i=1}^{3} s_ik_{1i}F_{2i}z_i + 2\sum_{i=1}^{3} s_ik_{1i}F_{4i}\left(2z_i^3 - 3r^2z_i\right) \\ \Phi &= 2\sum_{i=1}^{3} s_ik_{2i}F_{2i}z_i + 2\sum_{i=1}^{3} s_ik_{2i}F_{4i}\left(2z_i^3 - 3r^2z_i\right) \\ \sigma_r &= 2\sum_{i=1}^{3} e_iF_{2i} + 3\sum_{i=1}^{3} F_{4i}\left(-f_ir^2 + 4e_iz_i^2\right) - 2c_{66}G_0\frac{1}{r^2} + h_1r^2 \\ \sigma_\theta &= 2\sum_{i=1}^{3} e_iF_{2i} + 3\sum_{i=1}^{3} F_{4i}\left(-g_ir^2 + 4e_iz_i^2\right) - 2c_{66}G_0\frac{1}{r^2} + h_2r^2 \end{aligned} \tag{88}$$

$$\sigma_z = 2\sum_{i=1}^{3} a_i F_{2i} + 6\sum_{i=1}^{3} a_i F_{4i}\left(2z_i^2 - r^2\right) + h_3 r^2, \quad \tau_{rz} = -12\sum_{i=1}^{3} s_i a_i F_{4i} r z_i$$

$$D_r = -12\sum_{i=1}^{3} s_i c_i F_{4i} r z_i, \quad D_z = 2\sum_{i=1}^{3} c_i F_{2i} + 6\sum_{i=1}^{3} c_i F_{4i}\left(2z_i^2 - r^2\right) + h_4 r^2 \tag{89}$$

Substituting eqn (89) into eqn (86), we have

$$2r_0^2\sum_{i=1}^{3} e_i F_{2i} - r_0^2\sum_{i=1}^{3} F_{4i}\left(3f_i r_0^2 - h^2 s_i^2 e_i\right) + 2c_{66}G_0 + h_1 r_0^4 = 0$$

$$2r_1^2\sum_{i=1}^{3} e_i F_{2i} - r_1^2\sum_{i=1}^{3} F_{4i}\left(3f_i r_1^2 - h^2 s_i^2 e_i\right) + 2c_{66}G_0 + h_1 r_1^4 = 0$$

$$\sum_{i=1}^{3} s_i^2 a_i F_{4i} = 0$$

$$2\sum_{i=1}^{3} a_i F_{2i} + 3h^2\sum_{i=1}^{3} s_i^2 a_i F_{4i} = 0$$

$$6\sum_{i=1}^{3} a_i F_{4i} - h_3 = 0$$

$$2\sum_{i=1}^{3} c_i F_{2i} + 3h^2\sum_{i=1}^{3} s_i^2 c_i F_{4i} = d_3$$

$$6\sum_{i=1}^{3} c_i F_{4i} - h_4 = 0 \tag{90}$$

Solve eqn (90) successively, then F_{4i}, F_{2i} and G_0 will be obtained in turn. Substituting F_{4i}, F_{2i} and G_0 back into eqns (88) and (89) yields the solution of an annular piezoelectric rotating disk. Set $r_0 = 0$ and $G_0 = 0$, in eqn (90), then F_{4i} and F_{2i} can be calculated from eqn (90), that is, the solution for a solid piezoelectric rotating disk is obtained.

7.3. Solution for piezoelectric rotating circular shafts

The boundary conditions of a hollow piezoelectric rotating shaft are:

$$z = \pm\frac{h}{2}: \begin{cases} 2\pi \int_{2_0}^{r_1} \sigma_z r \, \mathrm{d}r = 0 \\ \tau_{rz} = 0 \\ 2\pi \int_{r_0}^{r_1} D_z r \, \mathrm{d}r = \pi\left(r_1^2 - r_0^2\right) d_0 \end{cases} \qquad r = r_k: \begin{cases} \sigma_r = 0 \\ \tau_{rz} = 0 \quad (k = 0, 1) \\ D_r = 0 \end{cases} \tag{91}$$

Set $F_{4i} = 0$ in eqns (88) and (89) and substitute eqn (89) into the boundary conditions eqn (91), then we have

$$4\sum_{i=1}^{3} a_i F_{2i} + h_3\left(r_1^2 + r_0^2\right) = 0$$

$$4\sum_{i=1}^{3} c_i F_{2i} + h_4\left(r_1^2 + r_0^2\right) = 2d_0$$

$$2r_0^2 \sum_{i=1}^{3} e_i F_{2i} - 2c_{66} G_0 + h_1 r_0^4 = 0$$

$$2r_1^2 \sum_{i=1}^{3} e_i F_{2i} - 2c_{66} G_0 + h_1 r_1^4 = 0 \tag{92}$$

Solve eqn (92) simultaneously, then we get F_{2i} and G_0. Substituting F_{2i} and G_0 back into eqns (88) and (89) and letting $F_{4i} = 0$ give the solution for a hollow piezoelectric rotating shaft. Let $r_0 = 0$ and $G_0 = 0$ in eqns (88) and (89), then F_{2i} can be calculated, that is, the solution for a piezoelectric rotating circular shaft is obtained. In the following sections we proceed to the equilibrium of a cone or a hollow cone, which is traction-free except at the apex. In Appendix B, the set-up of a coordinate system is described and the boundary conditions as well as the equilibrium equations of relevant forces are also listed.

8. Torsion problem of a cone loaded with concentrated force couple M_z at the apex

This is a free torsion problem of a cone, we take

$$\psi_i = 0, \quad (i = 1, 2, 3) \quad \text{and} \quad \psi_0 = \frac{A_0}{R_0} \tag{93}$$

where A_0 are unknown constants to be determined and

$$R_0 = \left(r^2 + s_0^2 z^2\right)^{1/2} \tag{94}$$

Substituting eqn (93) into eqns (2) and (4) leads to

$$u_r = w = \Phi = 0, \quad u_\theta = -A_0 \frac{r}{R_0^3} \tag{95}$$

$$\tau_{r\theta} = c_{66}\frac{3r^2}{R_0^5}A_0, \quad \tau_{\theta z} = c_{44}\frac{3s_0^2 rz}{R_0^5}A_0, \quad D_\theta = e_{15}\frac{3s_0^2 rz}{R_0^5}A_0$$

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = D_r = D_z = 0 \tag{96}$$

Substituting eqn (96) into eqn (B1a) indicates that eqn (B1a) has been satisfied. Constant A_0 can be determined by the global equilibrium condition eqn (B4c), i.e.,

$$M_z + \int_0^{2\pi}\int_0^{b\tan\alpha} r\tau_{r\theta}r\,\mathrm{d}r\,\mathrm{d}\theta = 0 \tag{97}$$

Substituting $\tau_{r\theta}$ in eqn (96) into eqn (97) gives

$$A_0 = M_z \Bigg/ \left[2\pi c_{66}\left(\frac{3\tan^2\alpha + 2s_0^2}{\left(\tan^2\alpha + s_0^2\right)^{3/2}} - \frac{2}{s_0}\right)\right] \tag{98}$$

When the cone reduces to a half-space, i.e., $\alpha = \pi/2$, then $A_0 = -M_z s_0/(4\pi c_{66})$.

9. Solution for a cone subjected to concentrated force P_z and point charge Q

This is an axisymmetric deformation problem, we take

$$\psi_0 = 0, \quad \psi_i = A_i \ln(R_i + z_i), \quad (i = 1, 2, 3) \tag{99}$$

where A_i $(i = 1, 2, 3)$ are unknown constants to be determined and

$$R_i = \left(r^2 + s_i^2 z^2\right)^{1/2}, \quad (i = 1, 2, 3) \tag{100}$$

Substituting eqn (99) into eqns (2) and (4) gives

$$u_r = \sum_{i=1}^{3} A_i \frac{r}{R_i(R_i + z_i)}, \quad w = \sum_{i=1}^{3} s_i k_{1i}\frac{A_i}{R_i}$$

$$\Phi = \sum_{i=1}^{3} s_i k_{2i}\frac{A_i}{R_i}, \quad u_\theta = 0 \tag{101}$$

$$\sigma_r = -2c_{66}\sum_{i=1}^{3} A_i \frac{1}{R_i(R_i + z_i)} - \sum_{i=1}^{3} n_i A_i \frac{z_i}{R_i^3}$$

$$\sigma_\theta = 2c_{66}\sum_{i=1}^{3} A_i \frac{1}{R_i(R_i + z_i)} - \sum_{i=1}^{3} m_i A_i \frac{z_i}{R_i^3}$$

$$\sigma_z = -\sum_{i=1}^{3} a_i A_i \frac{z_i}{R_i^3}, \quad \tau_{rz} = -\sum_{i=1}^{3} s_i a_i A_i \frac{r}{R_i^3}$$

$$D_r = -\sum_{i=1}^{3} s_i c_i A_i \frac{r}{R_i^3}, \quad D_z = -\sum_{i=1}^{3} c_i A_i \frac{z_i}{R_i^3}$$

$$\tau_{r\theta} = \tau_{\theta z} = D_\theta = 0 \tag{102}$$

The second equation of the boundary conditions on the cone surface, eqn (B1a), has been satisfied and the third and fourth equations can be deduced from the global equilibrium. Thus, only the first equation of the boundary condition eqn (B1a) and the following global equilibrium conditions need to be satisfied.

$$P_z + \int_0^{2\pi} \int_0^{b \tan \alpha} \sigma_z r \,\mathrm{d}r \,\mathrm{d}\theta = 0, \quad Q = \int_0^{2\pi} \int_0^{b \tan \alpha} D_z r \,\mathrm{d}r \,\mathrm{d}\theta \tag{103}$$

Substituting the relevant expressions in eqn (102) into eqn (103) and the first expression of eqn (B1a) gives

$$\sum_{i=1}^{3} \left(\frac{s_i}{H_i \tan \alpha} - 1 \right) a_i A_i = -P_z/(2\pi)$$

$$\sum_{i=1}^{3} \left(\frac{s_i}{H_i \tan \alpha} - 1 \right) c_i A_i = Q/(2\pi)$$

$$\sum_{i=1}^{3} \left(\frac{s_i a_i \tan \alpha}{H_i^3} - \frac{2c_{66}}{H_i N_i} - \frac{n_i s_i}{\tan \alpha H_i^3} \right) A_i = 0 \tag{104}$$

where

$$H_i = \sqrt{1 + s_i^2/\tan^2 \alpha}, \quad N_i = (H_i + s_i/\tan \alpha), \quad (i = 0,\ 1,\ 2,\ 3) \tag{105}$$

A_i can be obtained by solving eqn (104). When the cone reduces to a half-space, i.e., $\alpha = \pi/2$, we have

$$A_1 = \left[P_z(s_2 a_2 c_3 - s_3 a_3 c_2) + Q(s_2 a_2 a_3 - s_3 a_3 a_2) \right]/\Delta$$

$$A_2 = \left[P_z(s_3 a_3 c_1 - s_1 a_1 c_3) + Q(s_3 a_3 a_1 - s_1 a_1 a_3) \right]/\Delta$$

$$A_3 = \left[P_z(s_1 a_1 c_2 - s_2 a_2 c_1) + Q(s_1 a_1 a_2 - s_2 a_2 a_1) \right]/\Delta \tag{106}$$

where

$$\Delta = 2\pi \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ s_1 a_1 & s_2 a_2 & s_3 a_3 \end{vmatrix} \tag{107}$$

10. Bending problem of a cone loaded with concentrated force P_x at its apex

Take

$$\psi_0 = \frac{A_0 r \sin\theta}{R_0 + z_0}, \quad \psi_i = \frac{A_i r \cos\theta}{R_i + z_i}, \quad (i = 1, 2, 3) \tag{108}$$

where A_0 and $A_i (i = 1, 2, 3)$ are unknown constants to be determined.

Substituting eqn (108) into eqns (2) and (4) gives

$$u_r = \sum_{i=1}^{3} A_i \left(\frac{1}{R_i + z_i} - \frac{r^2}{R_i (R_i + z_i)^2} \right) \cos\theta - \frac{A_0 \cos\theta}{R_0 + z_0}$$

$$u_\theta = -\sum_{i=1}^{3} A_i \frac{\sin\theta}{R_i + z_i} + \left(\frac{1}{R_0 + z_0} - \frac{r^2}{R_0 (R_0 + z_0)^2} \right) A_0 \sin\theta$$

$$w = -\sum_{i=1}^{3} s_i k_{1i} \frac{A_i r \cos\theta}{R_i (R_i + z_i)}, \quad \Phi = -\sum_{i=1}^{3} s_i k_{2i} \frac{A_i r \cos\theta}{R_i (R_i + z_i)} \tag{109}$$

$$\sigma_r = 2c_{66} \frac{r \cos\theta}{R_0 (R_0 + z_0)^2} A_0 + 2c_{66} \sum_{i=1}^{3} A_i \frac{r \cos\theta}{R_i (R_i + z_i)^2} + \sum_{i=1}^{3} n_i A_i \frac{r \cos\theta}{R_i^3}$$

$$\sigma_\theta = -2c_{66} \frac{r \cos\theta}{R_0 (R_0 + z_0)^2} A_0 - 2c_{66} \sum_{i=1}^{3} A_i \frac{r \cos\theta}{R_i (R_i + z_i)^2} + \sum_{i=1}^{3} m_i A_i \frac{r \cos\theta}{R_i^3}$$

$$\sigma_z = \sum_{i=1}^{3} a_i A_i \frac{r \cos\theta}{R_i^3}, \quad D_z = \sum_{i=1}^{3} c_i A_i \frac{r \cos\theta}{R_i^3}$$

$$\tau_{r\theta} = c_{66} \sum_{i=1}^{3} A_i \frac{2r \sin\theta}{R_i (R_i + z_i)^2} + c_{66} \left(-\frac{2r \sin\theta}{R_0 (R_0 + z_0)^2} + \frac{r^3 \sin\theta}{R_0^3 (R_0 + z_0)^2} + \frac{2r^3 \sin\theta}{R_0^2 (R_0 + z_0)^3} \right) A_0$$

$$\tau_{rz} = \sum_{i=1}^{3} s_i a_i A_i \left(\frac{1}{R_i (R_i + z_i)} - \frac{z_i}{R_i^3} \right) \cos\theta + s_0 c_{44} \frac{\cos\theta}{R_0 (R_0 + z_0)} A_0$$

$$\tau_{\theta z} = \sum_{i=1}^{3} s_i a_i \frac{\sin\theta}{R_i (R_i + z_i)} A_i + s_0 c_{44} A_0 \left(\frac{1}{R_0 (R_0 + z_0)} - \frac{z_0}{R_0^3} \right) \sin\theta$$

$$D_r = \sum_{i=1}^{3} s_i c_i A_i \left(\frac{1}{R_i (R_i + z_i)} - \frac{z_i}{R_i^3} \right) \cos\theta + s_0 e_{15} \frac{\cos\theta}{R_0 (R_0 + z_0)} A_0$$

$$D_\theta = \sum_{i=1}^{3} s_i c_i \frac{\sin\theta}{R_i(R_i + z_i)} A_i + s_0 e_{15} A_0 \left(\frac{1}{R_0(R_0 + z_0)} - \frac{z_0}{R_0^3} \right) \sin\theta \tag{110}$$

Under the condition of traction-free cone surface, it can be proved without much difficulty that either the first or the second expression of the boundary condition eqn (B1a), is needed, if the following equation is required to hold in eqn (B2a)

$$P_x + \int_0^{2\pi} \int_0^{b\tan\alpha} (\tau_{rz}\cos\theta - \tau_{\theta z}\sin\theta) r \, \mathrm{d}r \, \mathrm{d}\theta = 0 \tag{111}$$

Substituting relevant expressions of eqn (110) into eqn (111) and expressions 1, 3 and 4 of eqn (B1a) leads to

$$\sum_{i=1}^{3} s_i a_i \left(1 - \frac{s_i}{H_i \tan\alpha} \right) A_i - c_{44} s_0 \left(1 - \frac{s_0}{H_0 \tan\alpha} \right) A_0 = \frac{P_x}{\pi}$$

$$\sum_{i=1}^{3} \left[\frac{2c_{66}}{H_i N_i^2} + \frac{n_i}{H_i^3} - s_i a_i \left(\frac{\tan\alpha}{H_i N_i} - \frac{s_i}{H_i^3} \right) \right] A_i + \left(\frac{2c_{66}}{H_0 N_0^2} - \frac{c_{44} s_0 \tan\alpha}{H_0 N_0} \right) A_0 = 0$$

$$\sum_{i=1}^{3} \left(\frac{s_i a_i}{H_i N_i} - \frac{s_i^2 a_i}{\tan\alpha H_i^3} - \frac{a_i \tan\alpha}{H_i^3} \right) A_i + \frac{c_{44} s_0}{H_0 N_0} A_0 = 0$$

$$\sum_{i=1}^{3} \left(\frac{s_i c_i}{H_i N_i} - \frac{s_i^2 c_i}{\tan\alpha H_i^3} - \frac{c_i \tan\alpha}{H_i^3} \right) A_i + \frac{e_{15} s_0}{H_0 N_0} A_0 = 0 \tag{112}$$

Hence, A_i can be obtained by solving eqn (112). When the cone reduces to a half-space, i.e., $\alpha = \pi/2$, we have

$$A_0 = -P_x / 2\pi s_0 c_{44}, \quad A_1 = \left[P_x (a_2 c_3 - a_3 c_2) \right] / \Delta$$

$$A_2 = \left[P_x (a_3 c_1 - a_1 c_3) \right] / \Delta, \quad A_3 = \left[P_x (a_1 c_2 - a_2 c_1) \right] / \Delta \tag{113}$$

where Δ is the same as eqn (107).

11. Bending problem of a cone subjected to concentrated force couple M_y at its apex

Take

$$\psi_0 = \frac{A_0 r \sin\theta}{R_0(R_0 + z_0)}, \quad \psi_i = \frac{A_i r \cos\theta}{R_i(R_i + z_i)} \quad (i = 1, 2, 3) \tag{114}$$

where A_0 and A_i ($i = 1, 2, 3$) are unknown constants to be determined.

Substituting eqn (114) into eqns (2) and (4) gives:

$$u_r = \sum_{i=1}^{3} A_0\left(\frac{z_i}{R_i^3} - \frac{1}{R_i(R_i+z_i)}\right)\cos\theta - \frac{A_0\cos\theta}{R_0(R_0+z_0)}$$

$$u_\theta = \sum_{i=1}^{3} \frac{A_i\sin\theta}{R_i(R_i+z_i)} + A_0\sin\theta\left(\frac{z_0}{R_0^3} - \frac{1}{R_0(R_0+z_0)}\right)$$

$$w = -\sum_{i=1}^{3} s_i k_{1i} A_i \frac{r\cos\theta}{R_i^3}, \quad \Phi = -\sum_{i=1}^{3} s_i k_{2i}\frac{r\cos\theta}{R_i^3}A_i \tag{115}$$

$$\sigma_r = -2c_{66}\sum_{i=1}^{3}\left(\frac{z_i}{R_i^3} - \frac{2}{R_i(R_i+z_i)}\right)\frac{\cos\theta}{r}A_i + \sum_{i=1}^{3} n_i\frac{3z_i r\cos\theta}{R_i^5}A_i$$

$$+2c_{66}\frac{A_0\cos\theta}{r}\left(\frac{2}{R_0(R_0+z_0)} - \frac{z_0}{R_0^3}\right)$$

$$\sigma_\theta = 2c_{66}\sum_{i=1}^{3}\left(\frac{z_i}{R_i^3} - \frac{2}{R_i(R_i+z_i)}\right)\frac{\cos\theta}{r}A_i + \sum_{i=1}^{3} m_i\frac{3z_i r\cos\theta}{R_i^5}A_i$$

$$-2c_{66}\frac{A_0\cos\theta}{r}\left(\frac{2}{R_0(R_0+z_0)} - \frac{z_0}{R_0^3}\right)$$

$$\sigma_z = \sum_{i=1}^{3} a_i\frac{3z_i r\cos\theta}{R_i^5}A_i$$

$$\tau_{r\theta} = 2c_{66}\sum_{i=1}^{3}\left(\frac{2}{R_i(R_i+z_i)} - \frac{z_i}{R_i^3}\right)\frac{\sin\theta}{r}A_i$$

$$+2c_{66}\frac{A_0\sin\theta}{r}\left(\frac{2}{R_0(R_0+z_0)} - \frac{z_0}{R_0^3}\right) - c_{66}\frac{3z_0 r\sin\theta}{R_0^5}A_0$$

$$\tau_{rz} = \sum_{i=1}^{3} A_i s_i a_i\frac{3r^2}{R_i^5}\cos\theta - \sum_{i=1}^{3} A_i s_i a_i\frac{\cos\theta}{R_i^3} + A_0 s_0 c_{44}\frac{\cos\theta}{R_0^3}$$

$$\tau_{\theta z} = \sum_{i=1}^{3} A_i s_i a_i\frac{\sin\theta}{R_i^3} + A_0 s_0 c_{44}\sin\theta\left(\frac{3r^2}{R_0^5} - \frac{1}{R_0^3}\right)$$

$$D_r = \sum_{i=1}^{3} A_i s_i c_i \frac{3r^2}{R_i^5} \cos\theta - \sum_{i=1}^{3} A_i s_i c_i \frac{\cos\theta}{R_i^3} + s_0 e_0 A_0 \frac{\cos\theta}{R_0^3}$$

$$D_\theta = \sum_{i=1}^{3} A_i s_i c_i \frac{\sin\theta}{R_i^3} + A_0 s_0 e_{15} \sin\theta\left(\frac{3r^2}{R_0^5} - \frac{1}{R_0^3}\right), \quad D_z = \sum_{i=1}^{3} c_i \frac{3z_i r\cos\theta}{R_i^5} A_i \tag{116}$$

If in the cone surface boundary condition eqn (B1a), $X_z^\alpha = 0$ and $D_n^\alpha = 0$ are required to hold and the following eqn (117) is also required to hold, then from eqn (B4b), it can be inferred that either $X_r^\alpha = 0$ or $X_\theta^\alpha = 0$ is needed.

$$M_y + \int_0^{2\pi}\int_0^{b\tan\alpha} \left[b(\tau_{rz}\cos\theta - \tau_{\theta z}\sin\theta) - r\sigma_z\cos\theta\right] r\,dr\,d\theta = 0 \tag{117}$$

From eqn (B2a), we have

$$\int_0^{2\pi}\int_0^{b\tan\alpha} (\tau_{rz}\cos\theta - \tau_{\theta z}\sin\theta) r\,dr\,d\theta = 0 \tag{118}$$

Then, eqn (117) can be simplified to

$$M_y - \int_0^{2\pi}\int_0^{b\tan\alpha} r^2\sigma_z\cos\theta\,dr\,d\theta = 0 \tag{119}$$

From eqn (119) as well as $X_r^\alpha = 0$, $X_z^\alpha = 0$ and $D_n^\alpha = 0$, the following system of equations in A_0, A_1, A_2 and A_3 is formed.

$$\sum_{i=1}^{3} a_i\left(\frac{s_i^2}{H_i^3\tan^3\alpha} - \frac{3s_i}{H_i\tan\alpha} - 2\right)A_i = \frac{M_y}{\pi}$$

$$\sum_{i=1}^{3}\left[2c_{66}\left(\frac{2}{H_iN_i} - \frac{s_i}{\tan\alpha H_i^3}\right) + \frac{3s_in_i}{\tan\alpha H_i^5} - \tan\alpha\left(\frac{3s_ia_i}{H_i^5} - \frac{s_ia_i}{H_i^3}\right)\right]A_i$$

$$+\left[2c_{66}\left(\frac{2}{H_0N_0} - \frac{s_0}{\tan\alpha H_0^3}\right) - s_0c_{44}\frac{\tan\alpha}{H_0^3}\right]A_0 = 0$$

$$\sum_{i=1}^{3}\left(\frac{s_ia_i}{H_i^5} - \frac{s_ia_i}{H_i^3} - \frac{3s_ia_i}{H_i^5}\right)A_i + \frac{s_0c_{44}}{H_0^3}A_0 = 0$$

$$\sum_{i=1}^{3}\left(\frac{s_ic_i}{H_i^5} - \frac{s_ic_i}{H_i^3} - \frac{3s_ic_i}{H_i^5}\right)A_i + \frac{s_0e_{15}}{H_0^3}A_0 = 0 \tag{120}$$

Consequently, A_i can be calculated from eqn (120).

12. Hollow cone problem

With respect to M_z torsion problem, A_0 can be calculated just by rewriting eqn (97) into the following form

$$M_z + \int_0^{2\pi} \int_{b\tan\beta}^{b\tan\alpha} r\tau_{r\theta} r \, dr \, d\theta = 0 \tag{97a}$$

As for the problem of P_z plus Q, take

$$\psi_0 = 0, \quad \psi_i = A_i \ln (R_i + z_i) + B_i \ln (R_i - z_i), \quad (i = 1, 2, 3) \tag{99a}$$

Then, from the global equilibrium of P_z and Q

$$P_z + \int_0^{2\pi} \int_{b\tan\beta}^{b\tan\alpha} \sigma_z r \, dr \, d\theta = 0, \quad Q = \int_0^{2\pi} \int_{b\tan\beta}^{b\tan\alpha} D_z r \, dr \, d\theta \tag{103a}$$

as well as $X_z^\alpha=0$, $D_n^\alpha=0$, $X_r^\alpha=0$, and $X_r^\beta = 0$, A_i and B_i can be determined.

In regard to the P_x bending problem, take

$$\psi_0 = \frac{A_0 r \sin\theta}{R_0 + z_0} + \frac{B_0 r \sin\theta}{R_0 - z_0}, \quad \psi_i = \frac{A_i r \cos\theta}{R_i + z_i} + \frac{B_i r \cos\theta}{R_i - z_i}, \quad (i = 1, 2, 3) \tag{108a}$$

From the global equilibrium condition of P_x

$$P_x + \int_0^{2\pi} \int_{b\tan\beta}^{b\tan\alpha} (\tau_{rz} \cos\theta - \tau_{\theta z} \sin\theta) r \, dr \, d\theta = 0 \tag{111a}$$

as well as $X_r^\alpha = 0$, $X_r^\beta = 0$, $X_\theta^\alpha = 0$, $X_z^\alpha = 0$, $X_z^\beta = 0$, $D_n^\alpha = 0$, and $D_n^\beta = 0$, constants A_i and B_i can be determined.

Regarding the M_y bending problem, take

$$\psi_0 = \frac{A_0 r \sin\theta}{R_0(R_0 + z_0)} + \frac{B_0 r \sin\theta}{R_0(R_0 - z_0)}$$

$$\psi_i = \frac{A_i \cos\theta}{R_i(R_i + z_i)} + \frac{B_i \cos\theta}{R_i(R_i - z_i)} \tag{114a}$$

From the global equilibrium condition of M_y

$$M_y - \int_0^{2\pi} \int_{b\tan\beta}^{b\tan\alpha} r^2 \sigma_z \cos\theta \, dr \, d\theta = 0 \tag{119a}$$

as well as $X_r^\alpha = 0$, $X_r^\beta = 0$, $X_\theta^\alpha = 0$, $X_z^\alpha = 0$, $X_z^\beta = 0$, $D_n^\alpha = 0$ and $D_n^\beta = 0$, constants A_i and B_i can be determined.

13. Conclusions

Due to material anisotropy and coupling between mechanical deformation and electric field, analytical solutions for piezoelectric materials are much more difficult to obtain and the process of solution is more complicated, compared with those in elasticity theory of isotropic materials. In general, stress components and displacements are dependent on material constants as shown in Table 2. However, in some solutions, stresses and electric displacements are independent of material constants, as shown in eqns (19), (25), (40) and (52). Furthermore, eqns (95), (96) and (98) show independence of piezoelectric constants and dielectric constants, yet dependence on elastic constants. All these results indicate that stress components in these solutions agree with those of the theory of elasticity for isotropic materials. Eqn (4) gives the stress components and electric displacements in terms of displacement functions, which automatically satisfy the equilibrium equations and Gauss equation provided that eqn (3) is satisfied. It should be noted that eqn (4) is different from eqn (6) of Ding et al. (1997a) in this sense. Making use of eqn (4) may bring convenience to the study of equilibrium problems. The analytical solutions obtained in the paper are also useful for the study of other problems relating to more complicated loads and boundary conditions by the superposition principle. Moreover, these solutions can serve as benchmarks for numerical methods such as the finite element method, the boundary element method, etc. All problems solved in the paper are listed in a table in Appendix C in order that they can be intuitively understood and conveniently applied.

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Appendix A: several harmonic function series

1. Harmonic polynomials for axisymmetric problems can be written in the following form:

$$\varphi_n(r, z) = z^n + \sum_{m=1}^{[n/2]} (-1)^m \frac{n(n-1), \ldots, (n-2m+1)}{2^{2m} m^2 (m-1)^2 \ldots 1} z^{n-2m} r^{2m} \tag{A1}$$

where $[n/2]$ denotes the largest integer $\leqslant (n/2)$. From eqn (A1), the first six harmonic polynomials can be written as follows:

$$\varphi_0(r, z) = 1, \quad \varphi_1(r, z) = z, \quad \varphi_2(r, z) = z^2 - \tfrac{1}{2} r^2, \quad \varphi_3(r, z) = z^3 - \tfrac{3}{2} r^2 z$$

$$\varphi_4(r, z) = z^4 - 3r^2 z^2 + \tfrac{3}{8} r^4, \quad \varphi_5(r, z) = z^5 - 5r^2 z^3 + \tfrac{15}{8} r^4 z \tag{A2}$$

2. A harmonic function series containing $\ln (r/r_1)$ (r_1 is a nonzero constant).

Another harmonic function series corresponding to $\varphi_n(r, z)$ that contains $\ln (r/r_1)$ is

$$\gamma_n(r, z) = \varphi_n(r, z) \ln \frac{r}{r_1} + Q_n(r, z) \tag{A3}$$

where

$$Q_n(r, z) = -\sum_{m=1}^{[n/2]} (-1)^m \frac{n(n-1), \ldots, (n-2m+1)}{2^{2m} m^2 (m-1)^2 \ldots 1} \left(\frac{1}{m} + \frac{1}{m-1} + \ldots + 1 \right) z^{n-2m} r^{2m} \tag{A4}$$

By eqns (A3) and (A4), the first six harmonic functions are written as follows:

$$\gamma_0(r, z) = \ln \frac{r}{r_1}, \quad \gamma_1(r, z) = z \ln \frac{r}{r_1}$$

$$\gamma_2(r, z) = \left(z^2 - \frac{1}{2}r^2\right) \ln\frac{r}{r_1} + \frac{r^2}{2}$$

$$\gamma_3(r, z) = \left(z^3 - \frac{3}{2}r^2 z\right) \ln\frac{r}{r_1} + \frac{3}{2}r^2 z$$

$$\gamma_4(r, z) = \left(z^4 - 3r^2z^2 + \frac{3}{8}r^4\right) \ln\frac{r}{r_1} + 3r^2z^2 - \frac{9}{16}r^4$$

$$\gamma_5(r, z) = \left(z^5 - 5r^2z^3 + \frac{15}{8}r^4 z\right) \ln\frac{r}{r_1} + 5r^2z^3 - \frac{45}{16}r^4 z \tag{A5}$$

3. It is not difficult to directly verify that the following functions are all harmonic functions

3.1. $$\frac{1}{R} \tag{A6}$$

where $R = \sqrt{r^2 + z^2}$

3.2. $$\ln(R+z) \text{ and } \ln(R-z) \tag{A7}$$

3.3. $$\frac{r\sin\theta}{R+z},\quad \frac{r\cos\theta}{R+z},\quad \frac{r\sin\theta}{R-z},\quad \frac{r\cos\theta}{R-z} \tag{A8}$$

3.4. $$\frac{r\sin\theta}{R(R+z)},\quad \frac{r\cos\theta}{R(R+z)},\quad \frac{r\sin\theta}{R(R-z)},\quad \frac{r\cos\theta}{R(R-z)} \tag{A9}$$

Appendix B: boundary conditions of a hollow cone

A hollow cone (the apex angle $2\alpha > 2\beta$) is considered. The origin of the coordinate system is taken to be the apex of the cone, and the z-axis be the common axis of the cone, which points into the cone. The xy-plane is parallel to the isotropic plane. Concentrated force $\mathbf{P} = P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}$, concentrated force couple $\mathbf{M} = M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k}$ and point charge Q are applied at the apex of the cone, where **i**, **j**, **k** are three unit vectors of a Cartesian coordinate system. Besides, the cone is loaded with surface forces: $\bar{X}_r^\alpha$, $\bar{X}_\theta^\alpha$, $\bar{X}_z^\alpha$, $\bar{X}_r^\alpha$, $\bar{X}_\theta^\beta$, $\bar{X}_z^\beta$ and prescribed electric displacements $\bar{D}_n^\alpha$ and $\bar{D}_n^\beta$.

In cylindrical coordinates, the boundary conditions on the surface are:

$$z/r = \cot\alpha:\quad \begin{aligned} &X_r^\alpha = \sigma_r\cos\alpha - \tau_{rz}\sin\alpha = \bar{X}_r^\alpha, \quad X_\theta^\alpha = \tau_{r\theta}\cos\alpha - \tau_{\theta z}\sin\alpha = \bar{X}_\theta^\alpha \\ &X_z^\alpha = \tau_{rz}\cos\alpha - \sigma_z\sin\alpha = \bar{X}_z^\alpha, \quad D_n^\alpha = D_r\cos\alpha - D_z\sin\alpha = \bar{D}_n^\alpha \end{aligned} \tag{B1a}$$

$$z/r = \cot\beta:\quad \begin{aligned} &X_r^\beta = \sigma_r\cos\beta - \tau_{rz}\sin\beta = \bar{X}_r^\beta, \quad X_\theta^\beta = \tau_{r\theta}\cos\beta - \tau_{\theta z}\sin\beta = \bar{X}_\theta^\beta \\ &X_z^\beta = \tau_{rz}\cos\beta - \sigma_z\sin\beta = \bar{X}_z^\beta, \quad D_n^\beta = D_r\cos\beta - D_z\sin\beta = \bar{D}_n^\beta \end{aligned} \tag{B1b}$$

Cut off a section of the cone by $z = b$ (constant) and the global equilibrium equations of this section are:

$$P_x + \int_0^{2\pi} \int_{b\tan\beta}^{b\tan\alpha} (\tau_{rz} \cos\theta - \tau_{\theta z} \sin\theta) r \, \mathrm{d}r \, \mathrm{d}\theta + \int_0^{2\pi} \int_0^b \left(\bar{X}_r^\alpha \cos\theta - \bar{X}_\theta^\alpha \sin\theta \right) z \, \mathrm{d}z \, \mathrm{d}\theta \tan\alpha / \cos\alpha$$

$$+ \int_0^{2\pi} \int_0^b \left(\bar{X}_r^\beta \cos\theta - \bar{X}_\theta^\beta \sin\theta \right) z \, \mathrm{d}z \, \mathrm{d}\theta \tan\beta / \cos\beta = 0 \quad \text{(B2a)}$$

$$P_y + \int_0^{2\pi} \int_{b\tan\beta}^{b\tan\alpha} (\tau_{rz} \sin\theta + \tau_{\theta z} \cos\theta) r \, \mathrm{d}r \, \mathrm{d}\theta + \int_0^{2\pi} \int_0^b \left(\bar{X}_r^\alpha \sin\theta + \bar{X}_\theta^\alpha \cos\theta \right) z \, \mathrm{d}z \, \mathrm{d}\theta \tan\alpha / \cos\alpha$$

$$+ \int_0^{2\pi} \int_0^b \left(\bar{X}_r^\beta \sin\theta + \bar{X}_\theta^\beta \cos\theta \right) z \, \mathrm{d}z \, \mathrm{d}\theta \tan\beta / \cos\beta = 0 \quad \text{(B2b)}$$

$$P_z + \int_0^{2\pi} \int_{b\tan\beta}^{b\tan\alpha} \sigma_z r \, \mathrm{d}r \, \mathrm{d}\theta + \int_0^{2\pi} \int_0^b \bar{X}_z^\alpha z \, \mathrm{d}z \, \mathrm{d}\theta \tan\alpha / \cos\alpha + \int_0^{2\pi} \int_0^b \bar{X}_z^\beta z \, \mathrm{d}z \, \mathrm{d}\theta \tan\beta / \cos\beta = 0 \quad \text{(B2c)}$$

$$Q = \int_0^{2\pi} \int_{b\tan\beta}^{b\tan\alpha} D_z r \, \mathrm{d}r \, \mathrm{d}\theta + \int_0^{2\pi} \int_0^b \bar{D}_n^\alpha z \, \mathrm{d}z \, \mathrm{d}\theta \tan\alpha / \cos\alpha + \int_0^{2\pi} \int_0^b \bar{D}_n^\beta z \, \mathrm{d}z \, \mathrm{d}\theta \tan\beta / \cos\beta \quad \text{(B3)}$$

$$M_x - \int_0^{2\pi} \int_{b\tan\beta}^{b\tan\alpha} \left[b(\tau_{\theta z} \cos\theta + \tau_{rz} \sin\theta) - r\sigma_z \sin\theta \right] r \, \mathrm{d}r \, \mathrm{d}\theta$$

$$- \int_0^{2\pi} \int_0^b \left(\bar{X}_\theta^\alpha \cos\theta + \bar{X}_r^\alpha \sin\theta - \bar{X}_z^\alpha \sin\theta \tan\alpha \right) z^2 \, \mathrm{d}z \, \mathrm{d}\theta \tan\alpha / \cos\alpha$$

$$- \int_0^{2\pi} \int_0^b \left(\bar{X}_\theta^\beta \cos\theta + \bar{X}_r^\beta \sin\theta - \bar{X}_z^\beta \sin\theta \tan\beta \right) z^2 \, \mathrm{d}z \, \mathrm{d}\theta \tan\beta / \cos\beta = 0 \quad \text{(B4a)}$$

$$M_y - \int_0^{2\pi} \int_{b\tan\beta}^{b\tan\alpha} \left[b(\tau_{\theta z} \sin\theta - \tau_{rz} \cos\theta) + r\sigma_z \cos\theta \right] r \, \mathrm{d}r \, \mathrm{d}\theta$$

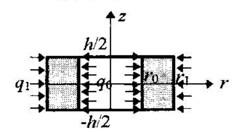
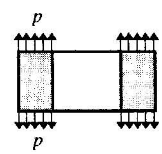
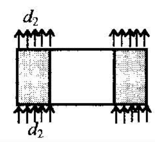
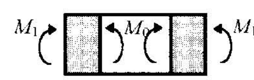
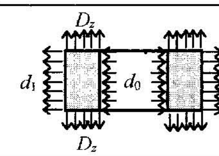
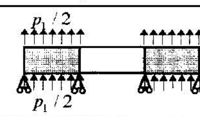
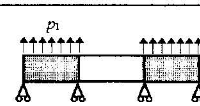
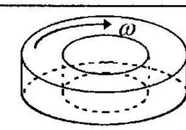
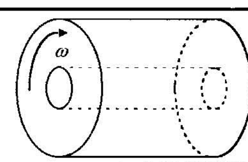
$$- \int_0^{2\pi} \int_0^b \left(\bar{X}_\theta^\alpha \sin\theta - \bar{X}_r^\alpha \cos\theta + \bar{X}_z^\alpha \cos\theta \tan\alpha \right) z^2 \, \mathrm{d}z \, \mathrm{d}\theta \tan\alpha / \cos\alpha$$

$$- \int_0^{2\pi} \int_0^b \left(\bar{X}_\theta^\beta \sin\theta - \bar{X}_r^\beta \cos\theta + \bar{X}_z^\beta \cos\theta \tan\beta \right) z^2 \, \mathrm{d}z \, \mathrm{d}\theta \tan\beta / \cos\beta = 0 \quad \text{(B4b)}$$

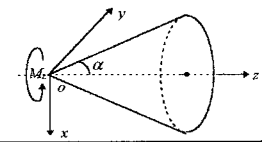
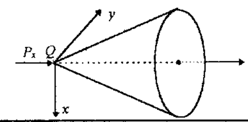
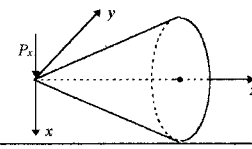
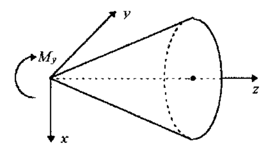
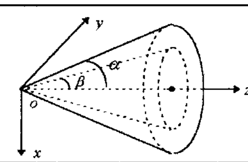
$$M_z + \int_0^{2\pi} \int_{b\tan\beta}^{b\tan\alpha} \tau_{\theta z} r^2 \, \mathrm{d}r \, \mathrm{d}\theta + \int_0^{2\pi} \int_0^b \bar{X}_\theta^\alpha z^2 \, \mathrm{d}z \, \mathrm{d}\theta \tan^2\alpha / \cos\alpha + \int_0^{2\pi} \int_0^b \bar{X}_\theta^\beta z^2 \, \mathrm{d}z \, \mathrm{d}\theta \tan^2\beta / \cos\beta = 0 \quad \text{(B4c)}$$

It is worth noting that in the first integral of eqns (B2a)–(B4a), $z = b$ and in the last two integrals $r = z \tan\alpha$ and $r = z \tan\beta$, respectively. As for the solid cone problem, let $\beta = 0$ in the above equations (2α is the apex angle of the cone) and eqn (B1b) does not need to be considered.

Appendix C: list of problems solved in this paper

No.	Problem	Location	Solution	Illustration
1	Rigid body displacement	Section 3	Eqn (9)	
2	Identical electric potential	Section 3	Eqn (10)	
3	Uniform radial compression	Section 4.1	Eqns (12), (17) and (19)	
4	Uniform axial tension	Section 4.2	Eqns (12), (20) and (22)	
5	Uniform axial electric displacement	Section 4.3	Eqns (12), (23) and (25)	
6	Pure bending	Section 5.1	Eqns (27), (39) and (40)	
7	Uniform radial electric displacements	Section 5.2	Eqns (27), (51) and (52)	
8	Annular plates under uniform axial loads on double surface	Section 6	Eqns (9),(54), (55), (58), (60) −(64), (66),(67) and (70)−(74)	
9	Annular plates under uniform axial loads on top surface	Section 6	Superposing the solution of No.8 with that of No.4 (where $p = p_1/2$)	
10	Rotating disks	Section 7	Eqns (88), (89), and (90)	
11	Rotating circular shafts	Section 7	Eqns (88), (89) (where $F_{4i} = 0$), and (92),	

continued

No.	Problem	Location	Solution	Illustration
12	Cone loaded with concentrated force couple M_z	Section 8	Eqns (95), (96), and (98)	
13	Cone subjected to concentrated force P_z and point charge Q	Section 9	Eqns (101), (102), and (106)	
14	Cone loaded with concentrated force P_x	Section 10	Eqns (109), (110) and (112)	
15	Cone loaded with concentrated force couple M_y	Section 11	Eqns (115), (116) and (120)	
16	Hollow cone	Section 12		

Note: A series of analytic solutions of the annular plate, circular plate, cylinder and hollow cylinder is obtained in this paper. To save space, we only list the annular plates and hollow cylinders in the above table. The solutions of the circular plate and the cylinder could be obtained by letting $r_0=0$, $B_0=B_{1i}=0$ and $G_0=G_{1i}=G_{3i}=0$ in the corresponding solutions of the annular plate and hollow cylinder.

Appendix D: nomenclature

a_i $(i=1, 2, 3)$	constants defined in eqn (5)
c_i $(i=1, 2, 3)$	constants defined in eqn (5)
c_{ijkl}	elastic stiffness constants
c_{11}, c_{12}, c_{13}, c_{33}, c_{44}	independent elastic stiffness constants of the transversely piezoelectric media
d_0, d_1	uniform radial electric displacement loaded on the inner and outer surfaces of the annular plate or the hollow cylinder and rotating shaft
d_2	uniform axial electrical displacement loaded on the annular plate or hollow cylinder
d_3	uniform axial electric displacement loaded on the rotating disk
D_i	components of electric displacement
$\bar{D}_n^\alpha$, $\bar{D}_n^\beta$	surface electric displacement loaded on the cone defined in eqn (B1a)
D_r, D_θ, D_z	components of electric displacement in the cylinder coordinate
e_i $(i=1, 2, 3)$	constants defined in eqn (14)

E_i	components of the electric field
e_{kij}	piezoelectric constants
e_{31}, e_{15}, e_{33}	independent piezoelectric constants of the transversely piezoelectric media
f_i $(i=1, 2, 3)$	constants defined in eqn (56)
F_i	body force
g_i $(i=1, 2, 3)$	constants defined in eqn (56)
h	thickness (height) of the annular plate (or hollow cylinder, rotating disk and rotating circular shaft)
H_i $(i=0, 1, 2, 3)$	functions defined in eqn (105)
k_{1i}, k_{2i} $(i=1, 2, 3)$	constants defined in eqn (41) in Ding et al. (1996a)
m_i $(i=1, 2, 3)$	constants defined in eqn (5)
M_x, M_y, M_z	moments along the positive directions of x-, y- and z-axes applied at the apex of the cone
M_1, M_2	bending moment loaded on the inner and outer sides of the annular plate or hollow cylinder
n_i $(i=1, 2, 3)$	constants defined in eqn (5)
N_i $(i=0, 1, 2, 3)$	functions defined in eqn (105)
p	uniform axial tension loaded on the annular plate or hollow cylinder
p_1	uniform axial loads on the annular plate simply-supported on the outer and inner surfaces
P_x, P_y, P_z	concentrated forces along the position directions of x-, y- and z-axes applied at the apex of the cone
q_0, q_1	uniform radial compression loaded on the inner and outer surfaces of the annular plate or hollow cylinder
Q	point charge applied at the apex of the cone
r, θ, z	coordinate axes in the cylinder coordinate $or\theta z$
r_0	inner radius of the annular plate (or hollow cylinder, rotating disk and rotating circular shaft)
r_1	outer radius of the annular plate (or hollow cylinder, rotating disk and rotating circular shaft)
R_0, R_i $(i=1, 2, 3)$	harmonic functions defined in eqn (94) and (100), respectively
$s_0=$	$\sqrt{c_{66}/c_{44}}$
s_i $(i=1, 2, 3)$	characteristic roots of a sixth-degree equation defined in eqn (32) in Ding et al. (1996), which satisfy Re $(s_i)>0$
u_i	components of displacement
u_r, u_θ, w	components of displacement in the cylinder coordinate
w_0	rigid body displacement
x, y, z	coordinate axes in the Cartesian coordinate $Oxyz$
$\bar{X}_r^\alpha$, $\bar{X}_\theta^\alpha$, $\bar{X}_z^\alpha$, $\bar{X}_r^\beta$, $\bar{X}_\theta^\beta$, $\bar{X}_z^\beta$	surface forces loaded on the cone defined in eqn (B1a)
z_i $(i=1, 2, 3)=$	$s_i z$
γ_n $(n=1, 2, 3\ldots)$	harmonic polynomials defined in eqn (A3)
2α	apex angle of the cone or outer apex angle of the hollow cone
2β	inner apex angle of the hollow cone
ε_{ij}	dielectric constants
$\bar{\varepsilon}_{ij}$	components of strain
ε_{11}, ε_{33}	independent dielectric constants of the transversely piezoelectric media
ρ	material density
ρ_f	density of free charges

σ_{ij}	components of stress
σ_r, σ_θ, σ_z, $\tau_{\theta z}$, τ_{zr}, $\tau_{r\theta}$	components of stress in the cylinder coordinate
Φ	electric potential
Φ_0	identical electric potential
φ_n $(n = 1, 2, 3\ldots)$	harmonic polynomials defined in eqn (A1)
ψ_0, ψ_i $(i = 1, 2, 3)$	displacement function
ω	angular velocity of the rotating disk or the rotating circular shaft

References

Chen, W.T., 1965. Stress in a transversely isotropic elastic cone under an asymmetric force at its vertex. ZAMP 16, 337–344.

Ding Haojiang, Chen Bo, Liang Jian, 1996a. General solution for coupled equations for piezoelectric media. Int. J. Solids and Structures 33 (16), 2283–2298.

Ding Haojiang, Chen Bo, Liang Jian, 1997a. On the Green's functions for two-phase transversely isotropic piezoelectric media. Int. J. Solids and Structures 34 (23), 3041–3057.

Ding Haojiang, Zou Daoqin, Ren Yongjian, 1995. Elasticity solutions of spherically isotropic cones under concentrated loads at apex. App. Mathematics and Mechanics 16 (11), 1025–1038.

Ding Haojiang, Liang Jian, Chen Bo, 1996b. Fundamental solution for transversely isotropic piezoelectric media. Science in China (Series A) 39 (7), 766–775.

Ding Haojiang, Wang Guoqing, Chen Weiqui, 1997b. General solution of plane problem of piezoelectric media expressed by 'Harmonic Functions'. App. Mathematics and Mechanics 19, 757–764.

Dunn, M.L., Wienecke, H.A., 1996. Green's functions for transversely isotropic piezoelectric media. Int. J. Solids and Structures 33 (30), 4571–4581.

Hu, H.C., 1953. On the three-dimensional problems of theory of elasticity of a transversely isotropic body. Acta Physica Sinica 9 (2), 130–144 (in Chinese).

Kogan, L., Hui, C.Y., Molkov, V., 1996. Stress and induction field of a spheroidal inclusion or a penny-shaped crack in a transversely isotropic piezoelectric material. Int. J. Solids and Structures 33 (19), 2719–2737.

Lee, J.S., Jiang, L.Z., 1996. Exact electroelastic analysis of piezoelectric laminae via state space approach. Int. J. Solids and Structures 33 (7), 977–990.

Lekhniskii, S.G., 1981. Theory of Elasticity of an Anisotropic Elasticity Body. Mir, Moscow.

Love, A.E.H., 1994. A Treatise on the Mathematical Theory of Elasticity, 4th ed. Dover, New York.

Luré, A.I., 1964. Three-dimensional Problems of the Theory of Elasticity. Interscience, New York.

Sosa, H.A., Castro, M.A., 1993. Elastroelastic analysis of piezoelectric laminated structures. Appl. Mech. Rev. 46 (11), s21–s28.

Sosa, H.A., Castro, M.A., 1994. On concentrated loads at the boundary of a piezoelectric half-plane. J. Mech. Phys. Solids 42 (7), 1105–1122.